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# Numerical Simulation of the Quantum Liouville–Poisson System

NAM-DUK SUH AND MARC R. FEIX

*PMMS/CNRS, Université d'Orléans, Orléans, France*

AND

PIERRE BERTRAND

*Physique Théorique, Université de Nancy I, Nancy, France*

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We simulate the one-dimensional quantum Liouville–Poisson system using the splitting scheme and the accompanying double Fourier transformation in $x$ and $p$ space. This code is used to study the quantum effects in the one-dimensional electrostatic plasma, i.e., the well-known nonlinear Landau damping and two stream instability problems. 

## 1. INTRODUCTION

The Wigner function introduced by E. Wigner [1] to calculate the quantum correction to the thermodynamical equilibrium was used successfully to study the theory of the quantum plasma by Klimontovich and Silin [2]. Even though the fact that it can have negative values makes a simple probabilistic interpretation of the Wigner function difficult [3], it is widely accepted that it can be used as a convenient mathematical tool to obtain the quantum corrections to the classical results [4].

In this paper we use the quantum Liouville equation–which is the time evolution equation of the Wigner function—in connection with the Poisson equation to see how the quantum effects change the collective plasma oscillations and especially how they change the phase space evolution of the one-dimensional electrostatic plasma. In this paper, we will consider only the quantum diffraction effect, i.e., we will not take into acount the second important quantum effect, the spin.

Because one cannot yet solve this quantum Liouville–Poisson system, we resort to the numerical simulation. Although particle codes are currently used in various classical problems, they are not well suited to look at the phase space evolution because of the limited number of particles used. In the quantum case, even the conception of particle codes is difficult to imagine and in this paper we use an Eulerian



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code which integrates the quantum Liouville equation directly in phase space. The method used is the well-known splitting scheme [5, 6]. We will show that if we separate the $x$ and $p$ dynamics, then it can be easily solved by applying the double Fourier transformation in $x$ and $p$, respectively. We compare the results with a linear theory to assure the validity of the simulation method.

Section 2 describes the quantum Liouville equation and the associated linear theory. In Section 3 we present the numerical method. In Section 4 we show the simulation results for the nonlinear Landau damping and the two stream instability problems. In Section 5 we conclude.

## 2. Quantum Liouville Equation and Linear Theory

### 2.1. *Quantum Liouville Equation*

Although the obtention of this equation is given in many papers, we sketch the main steps for the completeness of this paper. The Wigner distribution function corresponding to the pure quantum states is defined as

$$f_w(x, p, t) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} \psi^* \left( x - \frac{\Delta}{2}, t \right) \psi \left( x + \frac{\Delta}{2}, t \right) \exp\left( -i \frac{p\Delta}{\hbar} \right) d\Delta, \tag{1}$$

where the wave function $\psi$ satisfies the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \phi \Psi, \tag{2}$$

$x$ is the spatial position, $p$ the conjugate momentum, and $\hbar = h/2\pi$ is the Planck's constant. From the definition (1) we can obtain the evolution equation of $f_w$ with the aid of (2) [4, 7]. This equation is

$$\frac{\partial f_w}{\partial t} + \frac{p}{m} \frac{\partial f_w}{\partial x} + \frac{i}{2\pi\hbar} \iint_{-\infty}^{+\infty} \left\{ \phi \left( x - \frac{\Delta}{2}, t \right) - \phi \left( x + \frac{\Delta}{2}, t \right) \right\} \times \exp\left( i \frac{p - p'}{\hbar} \Delta \right) f_w(x, p', t)\, dp'\, d\Delta = 0. \tag{3}$$

The self-consistent potential $\phi$ is given by the Poisson equation

$$\frac{d^2\phi}{dx^2} = \int_{-\infty}^{+\infty} f_w\, dp - 1, \tag{4}$$

where we have put $\varepsilon_0 = n_0 = e = 1$ with $n_0$ the density of the motionless neutralizing background.

The most important difference between (3) and the classical Liouville equation is the nonlocality of the potential action. In the classical limit $\hbar \to 0$, the exponential

term in the integral kernel of (3) is violently oscillating and only a very small region around $\Delta = 0$ gives a significant contribution. Consequently we can use the approximation

$$\phi\left(x - \frac{\Delta}{2}, t\right) - \phi\left(x + \frac{\Delta}{2}, t\right) \simeq -\Delta \frac{\partial \phi}{\partial x} = E(x, t)\, \Delta.$$

Then after a double integration in (3), we get the classical Liouville equation

$$\frac{\partial f}{\partial t} + \frac{p}{m}\frac{\partial f}{\partial x} + E\frac{\partial f}{\partial p} = 0. \tag{5}$$

Hence (3) may be regarded as the quantum Liouville equation.

2.2. *Quantum Dispersion Relation*

All the linear phenomena of the classical plasma are completely described by the dielectric constant calculated by linearizing (5). Following the same procedure as the classical case, we can calculate the quantum dielectric constant which plays the same role as the classical one. We follow the demonstration given in [7].

We linearize (3) putting $f_w = F(v) + f_1(x, v, t)$, where $f_1$ is considered as a small perturbation around a homogeneous $F(v)$. It must be pointed out that because of the nonlinear relation connecting $\psi$ and $f_w$, it is impossible to consider the strictly equivalent linear problem in the Schrödinger formalism.

After linearizing (3) we take the Fourier transform over $x$. Thus we obtain ($k$ is the Fourier conjugate of $x$)

$$\frac{\partial f_1}{\partial t} - ikvf_1 + \frac{i}{2\pi\hbar^2}\iint \exp\left(i\frac{p - p'}{\hbar}\Delta\right) \times \left[\exp\left(ik\frac{\Delta}{2}\right) - \exp\left(-ik\frac{\Delta}{2}\right)\right]\phi(k)\, F(p')\, dp'\, d\Delta = 0.$$

The integration over $\Delta$ gives two Dirac function $\hbar\,\delta(p + \hbar k/2 - p')$ and $\hbar\,\delta(p - \hbar k/2 - p')$. Subsequently, the integration over $p'$ can be done easily and we have

$$\frac{\partial f_1}{\partial t} - ikvf_1 + E(k, t)\frac{F\left(p + \frac{\hbar k}{2}\right) - F\left(p - \frac{\hbar k}{2}\right)}{\hbar k} = 0. \tag{6}$$

Equation (6) with the Poisson one gives, after a simple calculation, the quantum dielectric constant

$$\varepsilon(k, \omega) = 1 + \frac{m\omega_p^2}{k}\int \frac{F\left(v + \frac{\hbar k}{2m}\right) - F\left(v - \frac{\hbar k}{2m}\right)}{\hbar k(\omega - kv)}\, dv, \tag{7}$$

where $\omega_p^2 = n_0 e^2/\varepsilon_0 m$ is the plasma frequency.

The dispersion relation can be obtained by putting $\varepsilon(k, \omega) = 0$. Now supposing that $\hbar k/mv_{\text{th}}$ is small we do the Taylor development of $F$ in the integral kernel of (13). Neglecting the terms higher than $(\hbar k/mv_{\text{th}})^3$, we obtain

$$1 + \frac{\omega_p^2}{\omega k} \int \frac{F'}{1 - kv/\omega}\, dv + \frac{\hbar^2 k}{24m^2} \frac{\omega_p^2}{\omega} \int \frac{F'''}{1 - kv/\omega}\, dv = 0. \tag{8}$$

The first two terms are the familiar classical terms which result in the linear dispersion relation $\omega^2 = \omega_p^2 + 3k^2 v_{\text{th}}^2 + 6k^4 v_{\text{th}}^4/\omega_p^2$, where $v_{\text{th}}$ is the thermal velocity of the system. The third one contains the quantum correction to the linear plasma oscillation. With the binomial development of $(1 - kv/\omega)^{-1}$ and following the traditional procedure, we arrive at the desired relation:

$$\omega^2 = \omega_p^2 + 3k^2 v_{\text{th}}^2 + \frac{6k^4 v_{\text{th}}^4}{\omega_p^2} + \frac{\hbar^2 k^4}{4m^2}. \tag{9}$$

From (9), we can give a little analysis about the magnitude of the quantum correction. Equation (9) can be rewritten in the form $\omega^2/\omega_p^2 = 1 + 3k^2\lambda_D^2 + 6k^4\lambda_D^4 + H^2 k^4 \lambda_D^4/4$, where $\lambda_D = v_{\text{th}}/\omega_p$ is the Debye length and $H = \hbar/mv_{\text{th}}\lambda_D$.

In this form, it is easy to see that the quantum correction can be greater than the classical $k^4$ correction. A priori, the validity of the expansion leading to Eq. (9) requests that $\hbar k/mv_{\text{th}} \ll 1$ and forbids the quantum correction to be larger than the classical $k^2$ term.

Nevertheless, the cold plasma case ($v_{\text{th}} \to 0$) can be handled directly (without any expansion) in Eq. (8) assuming $F(p) = \delta(p)$ and gives the relation

$$\omega^2 = \omega_p^2 + \hbar^2 k^4/4m^2.$$

We see consequently, that with a small $v_{\text{th}}$, the quantum correction can be the dominant one.

One can calculate the quantum correction term also by different method. It was shown in a little bit different language in [8]. We start from the linearized Schrödinger equation. Putting $\phi = \phi_1$, $\psi = 1 + \psi_1$ into (2) and keeping only the first-order term, we obtain

$$i\hbar \frac{\partial \psi_1}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_1}{\partial x^2} + \phi_1; \tag{10}$$

$\psi_0 = 1$ corresponds to an unperturbed $F(p) = \delta(p)$.

Assume that $\psi_1 \sim \exp(i(\omega t - kx))$. Then from (10) and its complex conjugate form we obtain

$$\psi_1 = -\phi_1 \Big/ \left(\hbar\omega + \frac{\hbar^2 k^2}{2m}\right), \qquad \psi_1^* = \phi_1 \Big/ \left(\hbar\omega - \frac{\hbar^2 k^2}{2m}\right).$$

Put these two terms into the relation $n_1(k, \omega) = \psi_1(k, \omega) + \psi_1^*(k, \omega)$. Thus

$$n_1(k, \omega) = \phi_1 \frac{\hbar^2 k^2/m}{\hbar^2\omega^2 - \hbar^2 k^4/4m^2}. \tag{11}$$

From (11) and the Fourier transformed form of the Poisson equation, we obtain easily

$$\omega^2 = \omega_p^2 + \frac{\hbar^2 k^4}{4m^2}. \tag{12}$$

Comparing (9) and (12), we see that for long wavelengths, the quantum and thermal corrections add independently of each other.

Thus the quantum correction term calculated by two different methods turns out to be the same. This can assure the validity of the quantum dielectric constant formalism.

The quantum correction is a minor one for small $k$, but for reasonable $k$ and $\hbar$, the effect becomes non-negligiable. But practically, we will have difficulty having $k$ small enough (for the validity of the long wavelength approximation) and $\hbar$ large enough (to detect the quantum correction). Having used too many approximations in deducing (9) the frequency calculated from (9) must not be quite exact in reality. So to compare the simulation results with the theoretical values, we prefer to calculate the theoretical $\omega$ directly from (7). We will come back to this point in Section 4.

## 3. Numerical Method

We want to solve the system (3), (4) for periodic boundary conditions in $x$. The main point of the code is the use of the splitting scheme and the accompanying double Fourier transformation successively and respectively in $x$ and $p$ directions. This scheme was used successfully to simulate the classical Vlasov–Poisson system and is second order in $\Delta T$ [5, 9].

Rather than solving Eq. (3) as a whole, this scheme splits the equation into two parts, the free particle part and the interaction part. Now assume that the potential is given as a Dirac function in time. See [5] for a discussion of the connexion between this ansatz and the splitting scheme,

$$\phi^*(x, t) = \phi(x, t)\, \Delta T \Sigma_n\, \delta(t - t_n), \qquad \text{where} \quad t_n = (n + \tfrac{1}{2})\, \Delta T.$$

Denoting $t_n^-$ and $t_n^+$, respectively as the time before and after the Dirac pulse, the time interval $(n\,\Delta T, (n+1)\,\Delta T)$ can be divided into three steps:

(a) $(n\,\Delta T \leqslant t < t_n^-)$. We have the free particle movement

$$\frac{\partial f_w}{\partial t} + \frac{p}{m}\frac{\partial f_w}{\partial x} = 0$$

performing the Fourier transformation in $x$, we obtain the solution

$$f_w(k, v, t_n^-) = f_w(k, v, n\,\Delta T)\exp(-ikv\,\Delta T/2). \tag{13}$$

(b) $(t = t_n)$. This corresponds to the interaction part and we can neglect the free particle motion while the potential is working. Thus we have

$$\frac{\partial f_w}{\partial t} + \frac{i}{2\pi\hbar^2}\iint \exp\left(i\frac{p-p'}{\hbar}\Delta\right)\left[\phi\left(x-\frac{\Delta}{2}, t\right)-\phi\left(x+\frac{\Delta}{2}, t\right)\right] f_w(p')\,dp'\,d\Delta = 0$$

performing the Fourier transform in $p$ space ($\lambda$ is the Fourier conjugate of $p$), we obtain the solution

$$f_w(x, \lambda, t_n^+) = f_w(x, \lambda, t_n^-)\exp\left\{-\frac{i}{\hbar}\left[\phi\left(x+\frac{\hbar\lambda}{2}, t_n^-\right)-\phi\left(x-\frac{\hbar\lambda}{2}, t_n^-\right)\right]\Delta T\right\}. \tag{14}$$

At this point, we must make two remarks. The Fourier transform on $p$ for a given $x$ (i.e., applied in the frame of the splitting method) turns the treatment of the interaction terms into a problem nearly as simple as the classical view; this is really the method to treat the quantum Liouville equation. The nonlocal character of the quantum interaction problem appears in a very clear way. Although we enforce the concept of an entity present at a point $x$, $v$ of the phase space, the distribution function of these quasi-particles at a given $x$ interact with the entire fields. Even as a mathematical tool, the concept of quasi-particle is not tenable.

(c) $(t_n^+ < t \leqslant (n+1)\,\Delta T)$. This is again the free particle motion and we repeat step (a) starting from $f(x, v, t_n^+)$.

Thus repeating the successive shifts

(a) The free particle motion from $n\,\Delta T$ to $t_n$:

$$f_w(x, v, n\,\Delta T) \xrightarrow{FTx} f_w(k, v, n\,\Delta T) \xrightarrow{X} \exp(-ikv\,\Delta T/2)$$

$$\xrightarrow{IFTx} f_w(x, v, t_n^-);$$

(b) Dirac pulse at $t = t_n$:

$$f_w(x, v, t_n^-) \xrightarrow{FTv} f_w(x, \lambda, t_n^-)$$

$$\xrightarrow{X} \exp\left(-\frac{i}{\hbar}\left[\phi\left(x+\frac{\hbar\lambda}{2}, t_n^-\right)-\phi\left(x-\frac{\hbar\lambda}{2}, t_n^-\right)\right]\right)$$

$$\xrightarrow{IFTv} f_w(x, v, t_n^+)$$

(c) again step (a) from $t_n^+$ to $(n+1)\,\Delta T$.

We can follow the time evolution of $f_w$.

A brief dimensional analysis is necessary to simulate Eq. (14). In our simulation we normalize the length $x$ by the Debye length $\lambda_D$. So let $X = x/\lambda_D$. Since the dimension of $\lambda$ is $[\lambda] = [mv]^{-1}$, we put $\Lambda = \lambda m v_{\text{th}}$. Thus, $x + \hbar\lambda/2 = \lambda_D(X + (\Lambda/2)(\hbar/mv_{\text{th}}\lambda_D))$. Since the dimension of $\hbar$ is $[\hbar] = ML^2T^{-1}$, we can correctly normalize $\hbar$ by $mv_{\text{th}}\lambda_D$. In fact we always take $m = v_{\text{th}} = \lambda_D = \omega_p = 1$.

## 4. Simulation Results

### 4.1. *Simulation of the Landau Damping in the Nonlinear Regime*

With the code mentioned above, we are going to study how the quantum effects change the nonlinear Landau damping phenomenon. The initial condition is

$$f_w(x, v, 0) = F(v)[1 + \alpha \cos k_0 x],$$

where $F(v) = (2\pi)^{-1/2} \exp(-v^2/2)$, $\alpha$ characterizes the strength of the perturbation, and $k_0 = 2\pi/L$ with $L$ the system length. We know very well what happens classically [5, 10]. If $\alpha$ is small enough, then the electric field created is also very small and the convective term of the Vlasov equation $v\,\partial f/\partial x$ homogenizes it. So the electric field damps away. This is the linear Landau damping. But if the electric field excited is large enough such that the nonlinearity becomes important, then there appears a hole structure in the phase space. This hole structure makes possible an equilibrium between the convective term and the force term $E\,\partial f/\partial v$ that drives collective oscillation of the plasma, so after a transient time, the excited wave subsists during very long time without damping away. This is the nonlinear saturation of the Landau damping.

In Fig. 1 we present classical phase spaces to help the comparison. It shows the phase space evolution corresponding to $k_0 = 0.3$ and $\alpha = 0.05$. Only the upper part ($v > 0$) is represented using grey points and the region $f > 0.02$ is represented by completely black part. This allows us to concentrate our attention to the interesting region $v > 3$.

One observes the formation of a hole at the velocity approximately equal to the phase velocity $v_\phi = \omega_k/k$, where $\omega_k$ is the linearized frequency associated to the wave number $k$. This hole formation is the origin of the saturation of the Landau damping and seems to be the building block of the nonlinear theory for the one-dimensional electrostatic plasma.

The quantum version of the phase space is shown in Fig. 2. The simulation was done for the parameters $k_0 = 0.4$, $\alpha = 0.2$, and $\hbar = 8$. There are two figures under each time indication. The left one represents the positive part of $f_w$ and the right one corresponds to the negative $f_w$. We do not see any more the formation of a hole. As is clear from Eq. (14), the potential acts non-locally, so the concept of a trapping by one monochromatic wave looses its sense.

As a matter of fact, the convective term can be interpreted in a classical way as

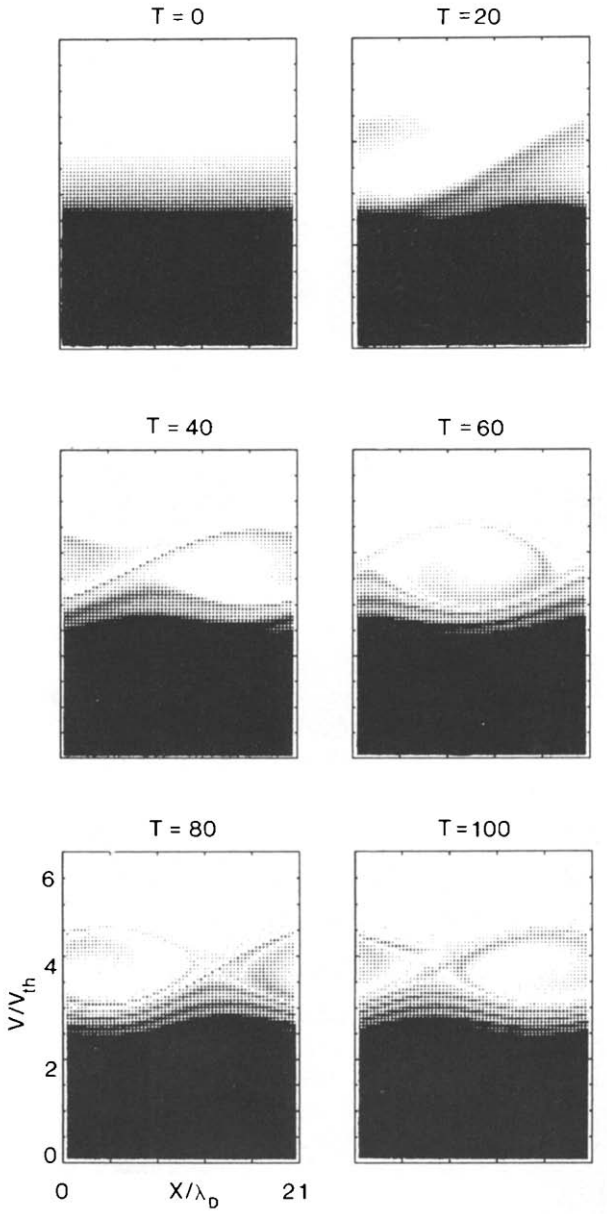


FIG. 1. Classical phase space evolution of the Landau damping. The black region is for $f \geqslant 0.02$ and $k_0\lambda_D = 0.3$, $\alpha = 0.05$.

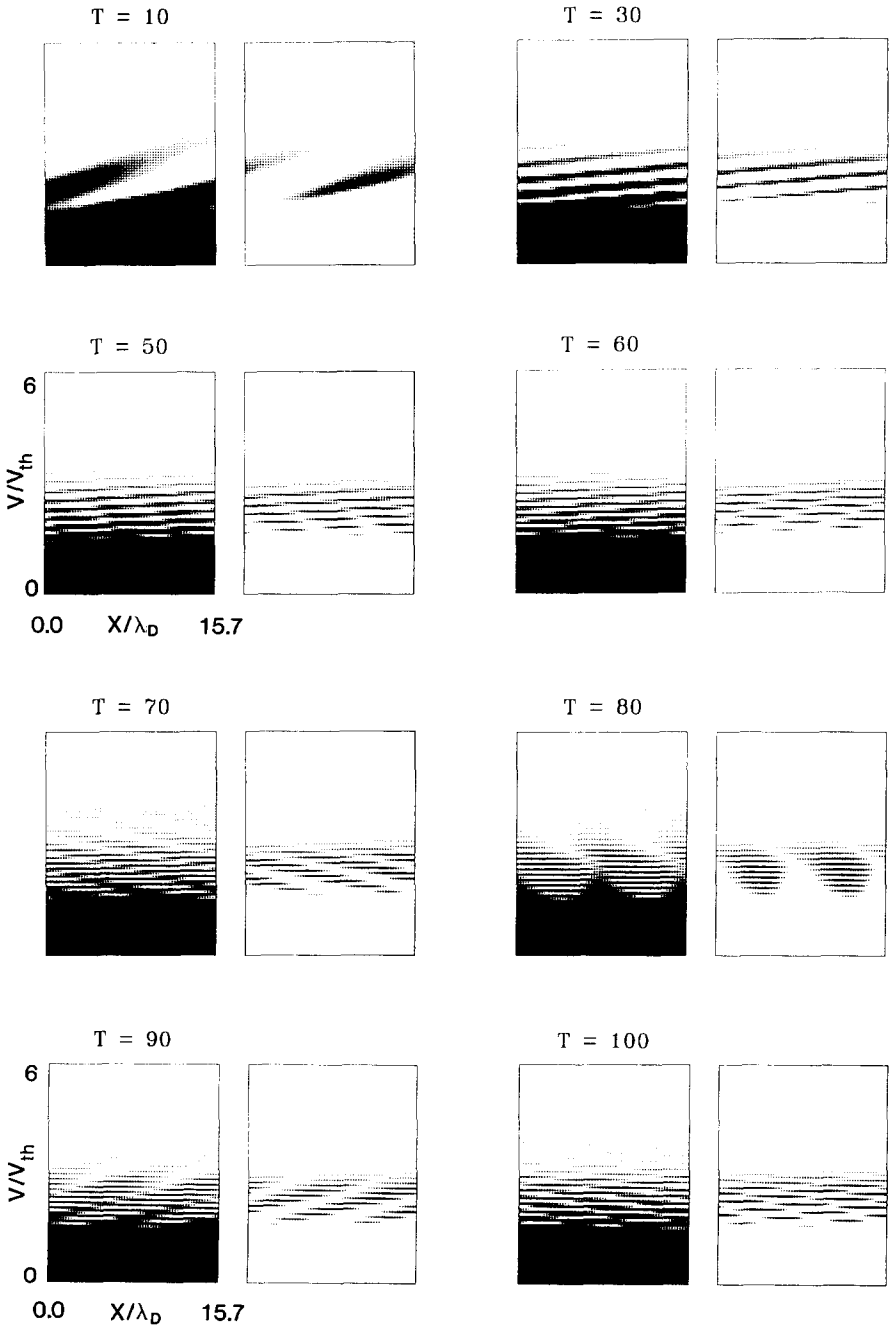


FIG. 2. Quantum phase space evolution for $\hbar = 8$. Of the two figures under each time indication, the left one represents the positive part of the Wigner function, and the right one shows the negative part. The black region is for $f \geq 0.02$ and $k_0 \lambda_D = 0.4$, $\alpha = 0.2$.

bringing homogenisation through the dispersion of the velocity of the quasi-particles. We see indeed that the horizontal strips which characterize the effect of this term become more and more filamented while the field goes on time-decreasing.

The validity of this simulation result must be backed by the comparison with the theory. Using the Fried and Conte function [11], we can calculate the collective oscillation frequency predicted by the linear theory from Eq. (7). Using the formula (9) we can get a quick evaluation, but it is too rough an approximation to be used. Our quantum dispersion relation is obtained by Eq. (7) putting $\varepsilon = 0$. Now insert $F(v) = (2\pi)^{-1/2} \exp(-v^2/2)$; then we obtain

$$Z\left\{\frac{1}{\sqrt{2}}\left(\frac{\omega}{k}+\frac{\hbar k}{2m}\right)\right\} - Z\left\{\frac{1}{\sqrt{2}}\left(\frac{\omega}{k}-\frac{\hbar k}{2m}\right)\right\} = \frac{\hbar k^3}{\omega_p^2}\sqrt{2}, \tag{15}$$

where

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{\exp(-x^2)}{x-\zeta}\, dx. \tag{16}$$

$Z$ can be calculated exactly and solving (15) for the complex $\omega = \omega_r + i\gamma$ we can obtain the frequency $\omega_r$ and the damping rate $\gamma$. Figure 3 shows the electric field evolution for $k_0 = 0.4$, $\alpha = 0.2$. The upper one is the classical result and the lower one is that corresponding to Fig. 2. It shows clearly that the hole formation is destroyed by the quantum effect and the linearized result is obtained. The frequency and the damping rate are measured as $\omega_r = 1.46\ \omega_p$ and $\gamma = -0.153$. The theoretical values calculated from (15) are $\omega_r = 1.489\ \omega_p$ and $\gamma = -0.1516$. The difference comes from the large perturbation $\alpha$ taken. We have done the same simulation with a smaller perturbation ($\alpha = 0.1$) and we obtain $\omega_r = 1.48\ \omega_p$ and $\gamma = -0.1526$ which agree very well with the theory. The classical frequency of Fig. 3 is $\omega = 1.24\ \omega_p$. The presence of the quantum effect is quite evident in this case.

We have found in the classical case, that the disappearance of the damping coincides with the formation of a hole. Such a hole may be visualized as a structure having a given volume in phase space. When $\hbar$ exceeds this volume, quantum mechanics forbids the existence of such a structure and consequently, we obtain an indirect check that indeed the saturation of Landau damping, in the classical case, was connected to the appearance of the hole.

### 4.2. *Simulation of the Two Stream Instability*

Our next simulation concerns the two stream instability. This is quite an old problem that has been abundantly studied theoretically and numerically. Its asymptotic state is a function of the energy only with phase space configuration characterized by a hole in the center ($v = 0$). This is called the BGK equilibrium with a space period $L$ and is a steady state solution of the Vlasov equation. Recent study showed that BGK equilibrium with more than one period, i.e., with more

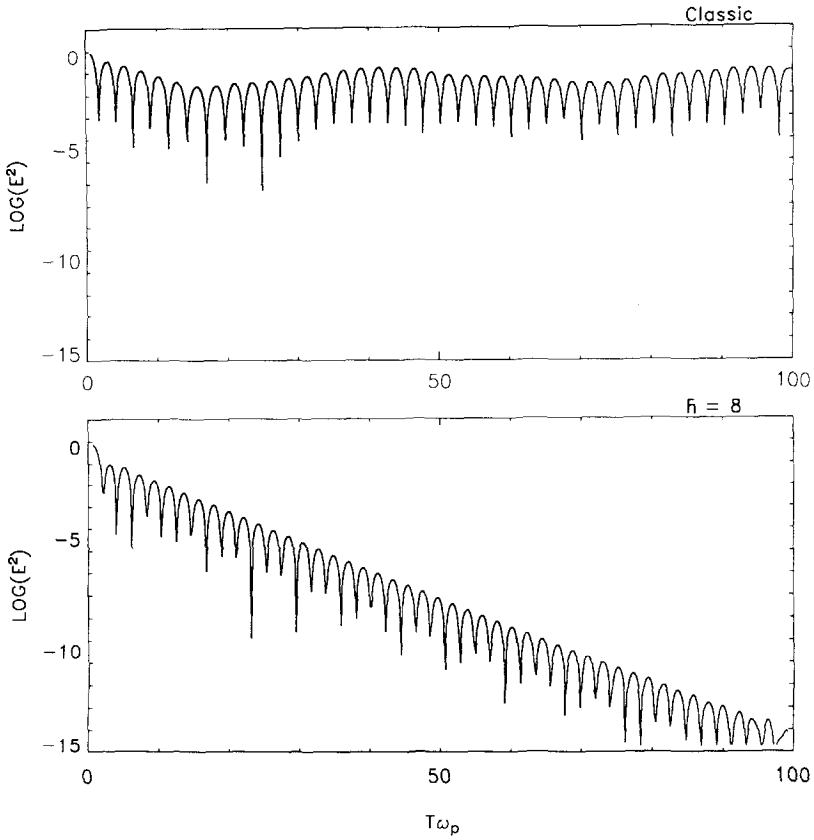


FIG. 3. Classical and quantum evolution of the electrostatic field for the Landau damping problem. $k_0\lambda_D = 0.4$, $\alpha = 0.2$.

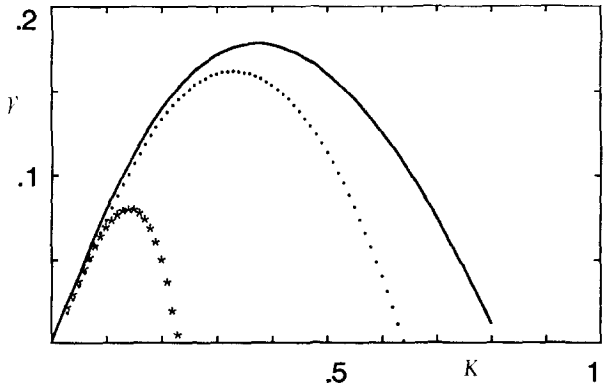


FIG. 4. Linear two stream instabiity growth rate as a function of the wavenumber $k$. The solid curve is the classical one. The dashed curve is for $\hbar = 2$ and the starred one is for $\hbar = 10$.

than one hole is unstable and tends toward the stable one hole structure by the coalescence between the holes [9].

The initial condition is

$$f_w(x, v, 0) = \frac{a}{\sqrt{\pi}}(1 + 5v^2)\exp(-v^2/2)(1 + \alpha \cos k_0 x), \qquad a = 1/6$$

with $\alpha = 1 * 10^{-3}$. The quantum dispersion relation is given by

$$\frac{\hbar k^3}{\omega_p^2} = \frac{a}{\sqrt{2}}\left(1 + 5\left(\frac{\omega}{k} + \frac{\hbar k}{2m}\right)^2\right) Z\left\{\frac{1}{\sqrt{2}}\left(\frac{\omega}{k} + \frac{\hbar k}{2m}\right)\right\} - \frac{a}{\sqrt{2}}\left(1 + 5\left(\frac{\omega}{k} - \frac{\hbar k}{2m}\right)^2\right) \times Z\left\{\frac{1}{\sqrt{2}}\left(\frac{\omega}{k} - \frac{\hbar k}{2m}\right)\right\} + \frac{5a\hbar k}{m}. \tag{17}$$

From Eq. (17) we can calculate the growth rate of the two stream instability for a given $k$ and $\hbar$. This is shown in Fig. 4. We observe that the quantum effect diminishes the growth rate and the wave number of marginal mode (the maximum wave number of the instability). We see that the classical marginal mode is $k_m = 0.816$ which results in the marginal wave length $L = 7.69$. For our simulation we choose the system length $L = 30.78$ and perturb over the fundamental mode $k_0 = 2\pi/L = 0.204$. From the $\gamma(k)$ curve (Fig. 4) we see that among the four unstable modes possible with this length system, the second one exhibits the largest growth rate, so we can expect that initially a 2-hole structure will appear.

Figure 5 shows the phase space evolution for the classical two stream instability. The initial two holes coalesce rapidly into one hole. Figure 8 a is the corresponding field energy. Figure 6 shows the quantum effect for $\hbar = 8$. The instability starts up at a much later time, or said otherwise, the linear state is prolonged much longer. This is clearly shown in the field evolution Fig. 8b. The growth rate obtained from Eq. (17) is $\gamma = 0.0757$. The simulation result Fig. 8b gives $\gamma = 0.071$. If we increase the quantum action $\hbar$ to 15, then the theory gives $\omega = 1.221\ \omega_p$ and $\gamma = -0.0029$ whcih means that there will be no instability and the plasma oscillates at the linear frequency. The simulation verifies this prediction. We pushed the calculation up to $\omega_p T = 200$, but Fig. 7 shows that really nothing happens. Figure 8c resembles a typical linear nearly undamped problem. We measure the frequency $\omega = 1.22\ \omega_p$ and $\gamma = -0.0025$. So they agree very well with the theoretical values.

We observed the same phenomena as in the preceeding Landau damping problem. The existence of an instability implies a hole, the volume of which is roughly speaking given by the product of the inverse wavenumber of maximum growth and the value of $p$ for which $f(p)$ is maximum. Again if $\hbar$ exceeds this value, then the instability does not appear. Of course, for long enough plasma, we can always have a hole, the volume of which is big enough to exceed $\hbar$, and an instability survives, but for longer and longer wavelengths. This explains the curves of Fig. 4.

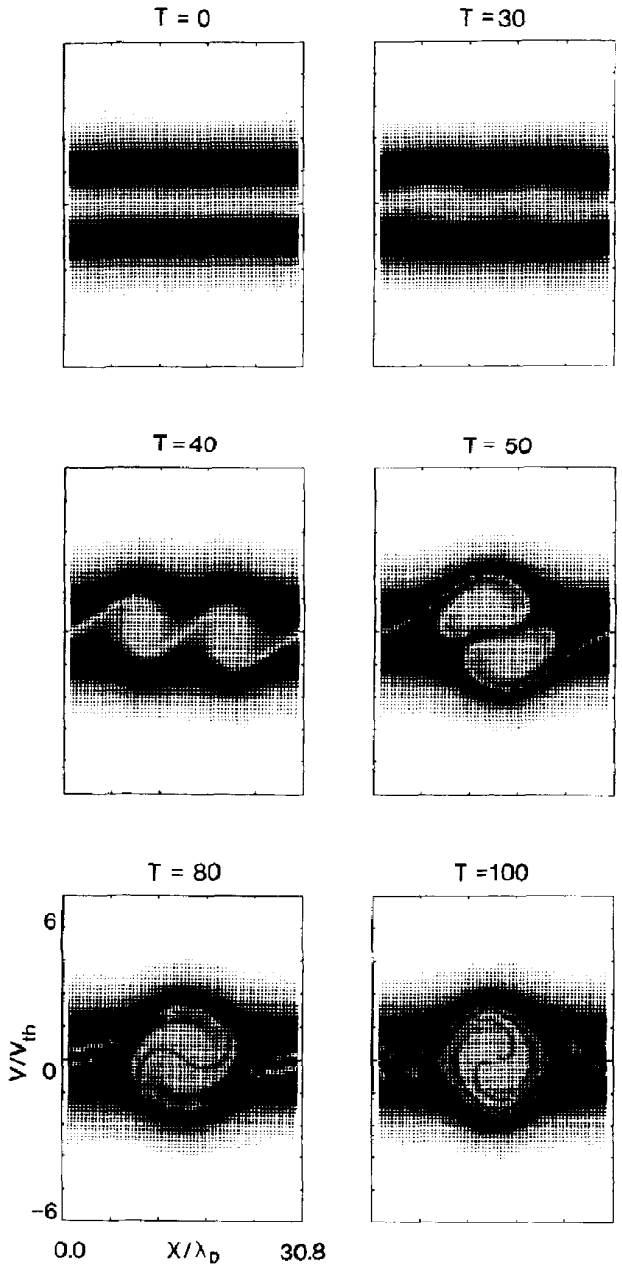


FIG. 5. Phase space evolution of the classical two stream instability.

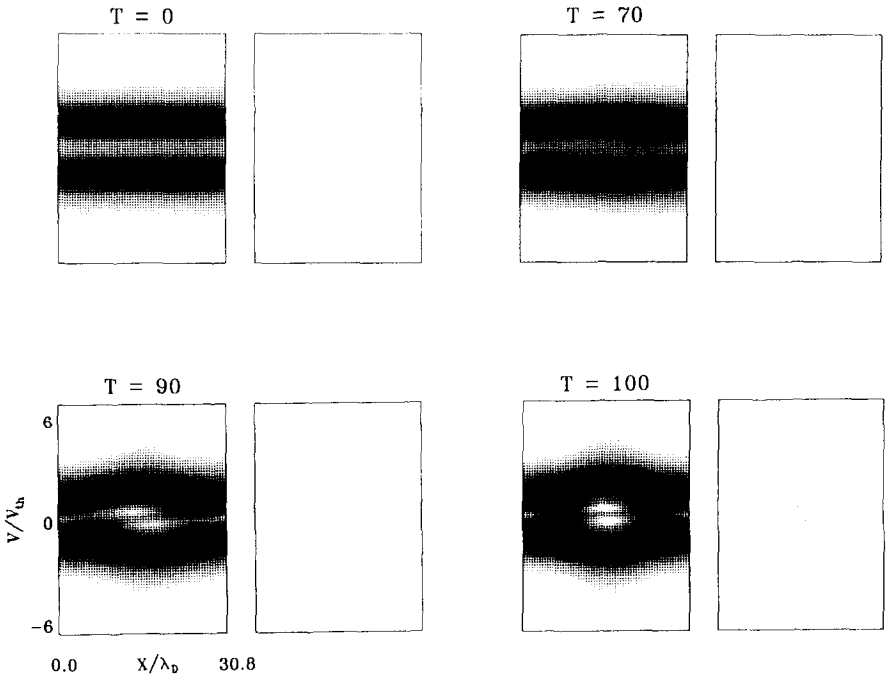


FIG. 6. Same as Fig. 5 for $\hbar = 8$. For most of the time $f$ remains positive and consequently the "negative frames" are empty. But at time $t = 100$, a negative value appears (two little segments of five points each).

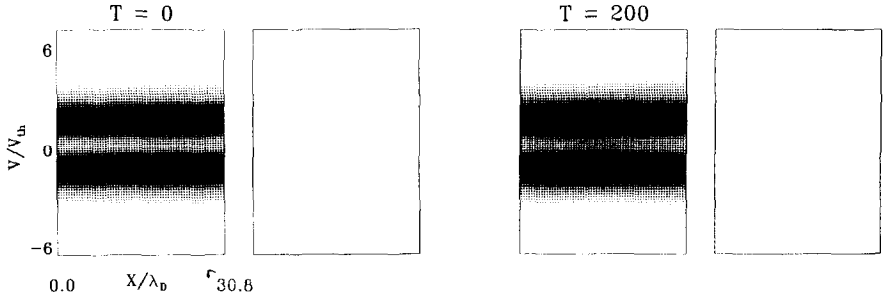


FIG. 7. Same as Fig. 5 for $\hbar = 15$. Note that the Wigner function is always positive.

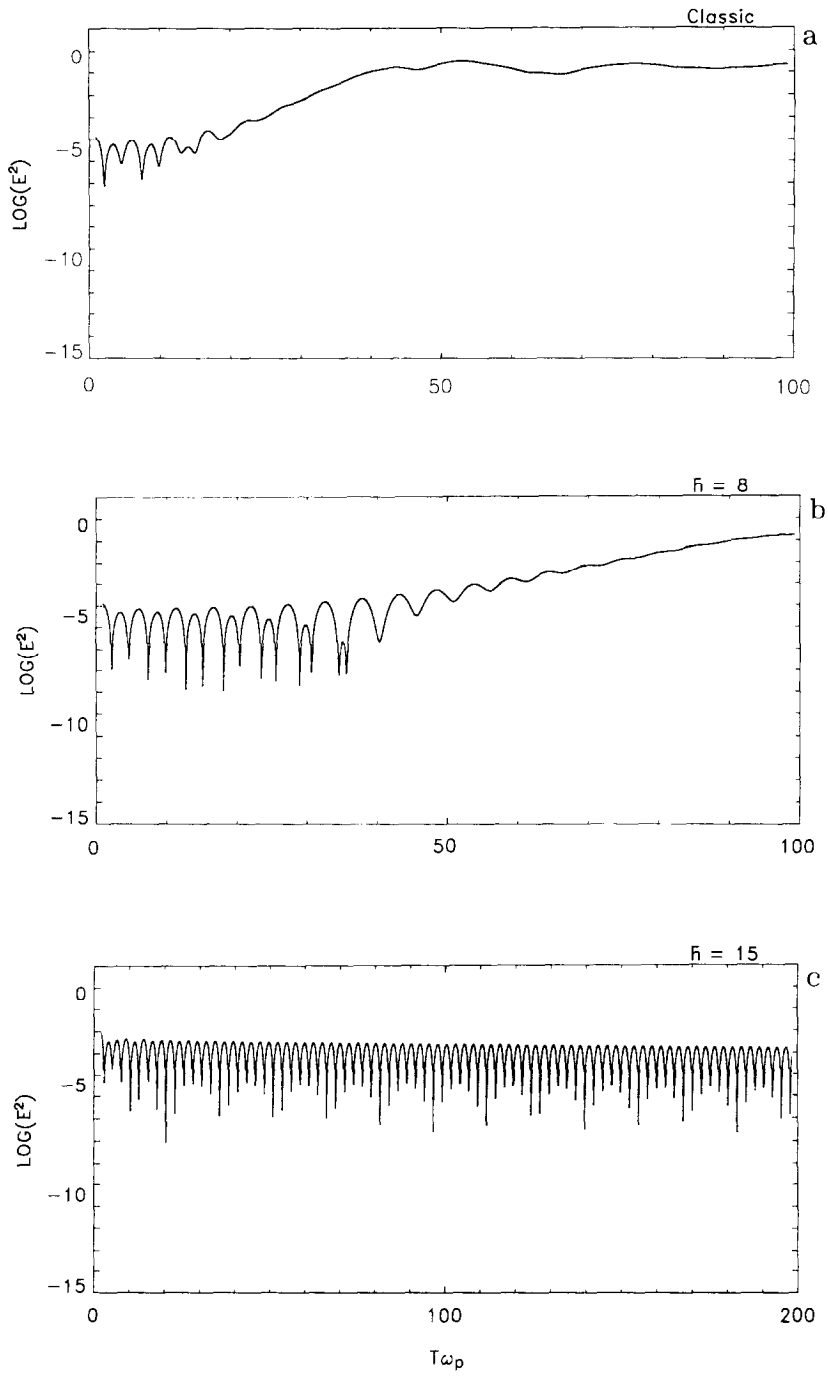


FIG. 8. Classical and quantum field evolution of the two stream instability.

## 5. Conclusion

In this article we have shown that the use of the splitting scheme and the accompanying double Fourier transformation in $x$ and $v$ space can successfully simulate the one-dimensional quantum Liouville equation. This code was used to study the quantum effects in the nonlinear Landau damping and the two stream instability problem. In the former problem we found that the quantum effect destroys the hole formation in phase space and consequently the nonlinear stabilization behaviour in the plasma disappears. If the quantum action $\hbar$ becomes larger, we get a strong linear damping behaviour. So the role of the hole in constructing nonlinear theory is shown to be very important. The simulation of the two stream instability confirms this result. For large $\hbar$, the instability starts at a later time and the linear behaviour is prolonged. For even larger $\hbar$, the quantum effect overcomes the instability and we obtain a gentle linear Landau damping.

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